MAKING SENSE OF STANDARD NOTATION
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ABSTRACT

In this paper I discuss the work of students who were introduced to standard combinatorial notation after they had already spent some time investigating concepts in combinatorics using their own personal representations. I show what strategies these students used to make sense of the standard notation and how they used this notation to express their ideas in general form. I note how the standard notation fulfilled an intellectual need: it allowed the students to express, in a concise and mathematically accurate form, generalizations of mathematical ideas that they had already developed.

THEORETICAL FRAMEWORK

It is considered helpful for students to develop their own personal notations when they are learning mathematics. The National Council of Teachers of Mathematics (NCTM) recommends (2000) that before students learn standard notation, they should first be afforded the opportunity to construct their own representations. According to Davis and Maher (1997), students who are provided with varied mathematical experiences build repertoires of representations, which they can use to deal with new mathematical ideas. Further, when students revisit problems, they refine their representations, moving from objects to symbols (Muter and Maher, 1999).

Nevertheless, personal representations are not adequate to deal with all mathematical situations. Standard notation provides a common language for communicating mathematically; it can also help students extract the important features of a mathematical problem, by highlighting what is significant as opposed to what is superficial (Lehrer, Schauble, Carpenter, and Penner, 2000). The NCTM notes that standard notation should be introduced, and teachers should decide when it is appropriate to introduce it. However, the NCTM does not offer specific advice about how to determine the appropriate time. But Skemp’s (1987) discussion of shortcomings of students’ personal notations suggests possible avenues for teachers to pursue. Skemp notes:

These ways of expression [personal notations] may often be lengthy, unclear, and differ between individuals. By experience of these disadvantages, and by discussion, children may gradually be led to the use of established mathematical symbolism in such a way that they experience its convenience and power for communicating and manipulating mathematical ideas (p. 183).

The work I describe here provides an illustration of students’ developing useful personal representations – notations that helped them to make sense of mathematical problems and to extend and generalize their knowledge. In
addition, they found that the standard notation valuable; it was convenient and powerful for expressing their mathematical ideas.

**DATA SOURCES**

This research uses data from a longitudinal study that followed a group of public school students from first grade through college (Maher, 2002). Data for this analysis (consisting of videotapes, student work, and researcher notes) are taken from after-school problem-investigation sessions with five students during two years of high school (1997 through 1999) and from individual interviews in 2002. This report focuses on the students’ work on two problems in combinatorics related to Pascal’s Triangle.

**PASCAL’S TRIANGLE**

If we start with a set of \( n \) objects and ask how many ways there are to select a subset consisting of \( r \) of those objects (\( r \) between 0 and \( n \)), we are working with combinations. The formula for combinations is:

\[
\binom{n}{r} = C(n,r) = \frac{n!}{r!(n-r)!}
\]

These numbers can be generated by an additive rule:

Equation 1: \( C(n,r) = C(n-1,r-1) + C(n-1,r) \)

The number \( C(n,r) \) is the \( r^{th} \) entry in the \( n^{th} \) row of Pascal’s Triangle, and the additive rule (Equation 1) is called Pascal’s Identity. Pascal’s Triangle is often written as shown in Figure 1. Figure 2 shows another way to write Pascal’s Triangle using one notation for combinations.

![Figure 1. Rows 0 to 4 of Pascal’s Triangle](image-url)
Figure 2. Rows 0 to 5 of Pascal’s Triangle in a notation for combinations

The answers to the combinatorics problems discussed below are found in Pascal’s Triangle.

**THE PROBLEMS**

The students worked on instances of the pizza problem and the towers problem. In general form, these are:

1. **Pizza:** How many pizzas is it possible to make when there are \( n \) toppings to choose from? Answer: each of the \( n \) toppings can be on or off the pizza; therefore, there are \( 2^n \) possible pizzas.

2. **Towers:** How many towers \( n \) cubes tall can be built when building towers from white and blue Unifix cubes? Answer: each position in the tower can be filled with either a blue cube or a white cube; therefore there are \( 2^n \) possible towers.

These problems are isomorphic; that is, there is a one-to-one correspondence between objects and relations in one problem and objects and relations in another problem (Powell, 2003). One specific form of the isomorphism is:

1. \( n \) = number of choices for pizza toppings = height of tower.
2. \( r \) = number of toppings on a given pizza = number of blue cubes in the tower.
3. \( C(n,r) \) = number of pizzas with exactly \( r \) toppings when there are \( n \) toppings to choose from = number of towers with exactly \( r \) blue cubes.

Figure 3 illustrates one instance of the isomorphism; it shows the relationship between a particular tower (the four-tall tower with two blue cubes in the middle) and a particular pizza (the sausage-pepperoni pizza).
I discuss below how the students, using personal representations, discovered the isomorphic relationship between the towers and pizza problems, as well as the rule governing Pascal's Identity. I also show how they were able to apply this understanding when they learned the standard notation, which they used to express Pascal's Identity in general symbolic form.

THE STUDENTS’ EARLY WORK

The students discussed here (Ankur, Brian, Jeff, Mike, and Romina) first worked on versions of the towers and pizza problems during elementary school. Early representations were concrete; for example, in order to answer the question of how many four-tall towers can be built from two colors, they built four-tall towers out of two colors of Unifix cubes. For the pizza problem, students’ initial representations were pictures of pizzas; an example is shown in Figure 4.

These early representations were realistic depictions of superficial features of the problems, but they did not help students extract the important mathematical features. The students moved on from drawings to letter codes; for example, the codes they used for the two pizzas show above were pl for the plain pizza and p for the pepperoni pizza. These representations were less attentive to irrelevant physical details, and they enabled the students to create locally organized lists. (For example they grouped pizzas in the “regular” group (single toppings) and “mixed” group (more than one topping.) However, these representations were not always easy to generalize.

THE STUDENTS’ LATER WORK

During their sophomore year in high school, the five students embarked on a further study of the pizza and towers problems, during after-school problem-solving sessions that took place about once every one to three months. They looked at how the problems are related — to each other, to the binomial
expansion, and to Pascal’s Triangle. In the process, they used their personal representations to explain individual instances of Pascal’s Identity and to show how the problems are related to Pascal’s Triangle and to each other. Finally, they made use of the standard notation to generalize their findings and to write Pascal’s Identity in standard notation. Three years after the final session, in individual interviews, three of the students (Mike, Ankur, and Romina) regenerated their earlier work. I discuss below the evolution of these events.

**EPISODE 1: A NEW REPRESENTATION**

During their first high school meeting, the students revisited the four- and five-topping pizza problems. Four of the students (Ankur, Brian, Jeff, and Romina) used a numerical code: for the four-topping case, the toppings were represented by 1, 2, 3, and 4. This enabled them to organize their enumeration of the 16 four-topping pizzas, and this representation system was not difficult to generalize. However, the students’ organizational system was not sufficiently powerful to handle the five-topping pizza problem; when it came to the 32 pizzas than can be made when there are five toppings to choose from, their list yielded only 30 pizzas. Further, this representational system did not provide a representation of the plain pizza.

Mike created a different representation, which was also easy to generalize. In addition, it handled cases that the others’ system did not provide for: it provided an easy way to represent the plain pizza, and it enabled Mike to demonstrate that there are 32 possible pizzas when there are five toppings to choose from. Moreover, it made explicit an important mathematical feature of the pizza problem: the fact that there are two choices for each topping (on or off the pizza). Mike’s representation system used the binary numbers 0 and 1. Mike made a table with columns that represented topping choices; he used 0 to represent a topping not on the pizza and 1 to represent a topping on the pizza. With this scheme, any \( n \)-digit binary number corresponds to one particular pizza in the list of possibilities when there are \( n \) toppings to choose from. For example, a plain pizza is represented by 0000 and one of the one-topping pizzas (for example, pepperoni) would be represented by 0010. Mike also noted that binary numbers could be used to represent answers to the towers problems. Mike explained this notation to his fellow students, and they added it to their toolkits of representations.

**EPISODE 2: INTRODUCTION TO COMBINATIONS**

At a meeting about a month after Mike created the binary representation, the group turned to a specific instance of the five-tall towers problem: when building towers from red and blue cubes, how many five-tall towers are there with exactly two blue cubes?

The researcher used this opportunity to discuss other questions related to five-tall towers – how many towers have exactly zero, one, three, four, and five
blue cubes. This discussion reminded some of the students that, back in middle school, they had organized their towers that way – according to number of cubes of a specific color. She showed how the answers to all these questions can be found in the fifth row of Pascal’s Triangle; that is, the numbers 1 5 10 10 5 1 gave, respectively, the number of five-tall towers with exactly 0, 1, 2, 3, 4, and 5 blue cubes. She showed the group how to express these numbers in the notation of combinations. (See row 5 of Figure 2.)

**EPISODE 3: AN EXPLANATION OF PASCAL’S IDENTITY USING TOWERS**

A month after Episode 2, three members of the group (Ankur, Jeff, and Romina) met with a visiting researcher. In the process of explaining their recent investigations to him, they discovered how Pascal’s Identity can be explained by the physical operation of building new towers from existing towers.

They started by telling the visitor that all the numbers in the binomial expansion, the pizza problems, and the towers problems were related, but they could not explicitly state the relationship. With some help, they recalled the previous month’s discussion about relating towers to Pascal’s Triangle. They were then asked to work out the relationship between pizzas and Pascal’s Triangle. Jeff started by suggesting that the leftmost 1 represented a plain pizza. They worked their way across the row using pizzas with one, two, three, four, and five toppings to represent the numbers 6, 15, 20, 15, and 6. They represented the final 1 by what they called the “supreme” pizza, the pizza with all possible toppings. Figure 5 is a diagram of their representations.

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1     6    15    20    15    6     1
plain   1 topping   2 toppings             ...             the “supreme”
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*Figure 5. The 6-topping pizzas and row 6 of Pascal’s Triangle*

At the end of the session, they returned to the towers representation in order to explain a specific instance of Pascal’s Identity (how to build row 6 starting from row 5). The visitor asked them to explain how the two 10’s in row 5 combine to make the 20 in row 6. The students explained that the first 10 counted the ten five-tall towers that have exactly two red cubes, and the second 10 counted the ten five-tall towers that have exactly three red cubes. So, they explained, one would add a red cube on top of the first ten towers and a blue cube on top of the other ten towers. Figure 6 illustrates their explanation.
The students were able to explain the generative rule for Pascal’s Triangle for specific numbers using their knowledge of a specific combinatorics problem, but their explanation required the listener to be aware of their referent.

**EPISODE 3: AN EXPLANATION OF PASCAL’S IDENTITY USING PIZZAS**

A few months after the above episode, Mike met with a researcher, and Mike told her how the pizza problem is related to Pascal’s Triangle. At the researcher’s request, Mike wrote down his thoughts and sent them via email. Below is a portion of Mike’s email in which he explained Pascal’s Identity in terms of pizzas. (The underlined numbers he mentioned are shown in Figure 7.)

In the 1331 sequence, the first position represents the binary number with all zeroes (no topping pizza) but when you add another topping, it could either have it or not. So now this 1 pizza combination turned into two. So do all the other combinations. … Why do we add? We add because of the fact that every combination will get another place (a 1 and a 0) therefore it doubles the amount of combinations. The reasons why we add numbers that are next to each other is simply because the underlined 1 will become into two new pizzas after another topping choice is added. One of those will be the same (no toppings) and the other will have one topping. The same will happen for the 2, the two will become 4, 2 the same (one topping) and two with two toppings. Now you have three one-topping pizzas 1 comes from the 1 in the “upper level” of the triangle, and two come from 2 in the “upper level.” That is why ya add ‘em…
Like his colleagues, Mike provided an explanation for a specific instance of Pascal’s Identity that required his correspondent to understand his specific referent. But there was generality in his answer, and he did not feel the need to mention specific pizza toppings (“this 1 pizza combination turned into two. So do all the other combinations”).

**EPISODE 5: WRITING PASCAL’S IDENTITY IN GENERAL FORM**

A few months after Mike sent his email, the students met for an extended evening session. Four students participated in the early part of the session (Ankur, Jeff, Mike, and Romina). First they reviewed the meaning of the numbers in Pascal’s Triangle using various representations. Recalling his previous work, Mike noted that row 3 of Pascal’s Triangle can be thought of as the number of possible pizzas when there are three toppings to choose from. He described how those pizzas can be used to build the pizzas in row 4. He specifically described what happens to the three pizzas in row 3 representing the one-topping pizzas. If you do not add the new topping, they become part of the four one-topping pizzas in row 4. If you do add the new topping, they become part of the six two-topping pizzas in row 4. A portion of the transcript follows. Mike’s procedure is illustrated in Figure 8.

Mike: Um, all right. If, all right, let's go to, let's go to this one. This would be like three different places I guess. [Mike points to the 1331 row of Pascal's triangle.] And um-

Jeff: Which one we looking at?

Mike: That one right there. [Mike points to 1331.] You have three-

Jeff: [Jeff inserts a reference to the binomial expansion.] That would be a plus b to the third. …

Mike: [Mike returns to the pizza representation.] So first category is everything with no toppings. [Mike points to the first 1 in the 1331 row and then to the first 3.] … There’s all the, the ones that have one topping. There’s your 3 choose 1. And there’s three different combinations you could put that. … But when you add another place, another topping-
Jeff: That could be one or the other, one or the other- …

Mike: So all these threes would either move up a step onto the next category and have two toppings [right arrow in Figure 8]. Or they might stay behind and still only have one [left arrow in Figure 8]. So three, three will get a topping, go to this one. And three won't, will stay.

![Figure 8. Doubling the number of pizzas](image)

Notice that even though they are discussing specific numbers in Pascal’s Triangle, they reference a general procedure – and they do not reference specific pizza toppings. It appears that the students have a notion of a general rule, but they express that idea in terms of specific cases.

After they provided the pizza explanation for row 3, the researcher asked them to work in the “choose” notation that Mike had already mentioned (in the transcript shown above), drawing row 3 in the notation shown in Figure 2. She asked them to illustrate the addition rule using that notation. They responded with the diagram shown in Figure 9, which Mike explained in terms of pizzas: “this guy [3 choose 1] gets another topping, I guess, so he would be a two. And this guy [3 choose 2] doesn't, so it stays two.”

![Figure 9. An instance of Pascal's Identity in “choose” notation](image)

Brian arrived during this discussion, and so the researcher asked the other four students to fill Brian in and also to generalize their results. The students responded with the diagram shown in Figure 10, which they rewrote as Equation 2. This is equivalent to Equation 1. The group had thus produced an equation that conformed to the “established mathematical symbolism” mentioned by Skemp. It was easily understandable, concise, and general.
When asked what the equation meant, Jeff referred to the pizza problem in general form, indicating that the first term stood for pizzas that would get a topping added and the second term stood for pizzas that would keep the same number of toppings. It seems plausible that the notation had freed the students from the necessity of working with specific numbers. Just as the use of the binary notation had freed the students from the necessity of referring to specific toppings, the use of the standard notation permitted Jeff to talk about general entries in Pascal’s Triangle rather than specific pizzas with specific numbers of toppings.

In the next sections, I discuss follow-up interviews with three of the students (Mike, Ankur, and Romina), when they were in college. (Note that they had not studied Pascal’s Triangle or combinatorics in college.) They continued to show an impressive ability to recall, extend, and explain their work in high school.

**EPISODE 6: INTERVIEW WITH MIKE**

About three years after Episode 5, when he was in the second year of college, Mike was interviewed by the original researcher. The researcher (R) mentioned Episode 5, showed Mike a copy of Figure 1, and then asked him to talk about Pascal’s Triangle. Mike responded with a description of row 2 of Pascal’s Triangle and then a generalization to following rows.

R: It was looking at how the triangle grew. And that was the question. How can you talk about how that triangle grows? And you had used the example of pizzas to think about...

Mike: OK. If you had no toppings, that would be one pizza.

R: OK. So where is that on the triangle?
Mike: Well, I'm going to just draw it and then we'll find it. You know. If you're using just one topping, you can make two possible pizzas with that. And then if you have all the toppings, that's one. Right. And then automatically, I see that, that relates to this row. [Mike points to row 2.] And I'm pretty sure it would go down, this is like a third topping, and a fourth topping. [Mike indicates rows 3 and 4.] Now I think the way I thought about it is, like, the row on the outside would be your plain pizza. [Mike refers here to the 1s down the left side of Pascal's Triangle.] And there's only one way to make a plain pizza. And the next … the next one over would be how many pizzas you could make using only one topping, and then so on until you get to the last row [Mike refers here to the last number in the row.] which is all your toppings. And, once again, you can only make one pizza out of that.

Notice that although Mike used the pizza problems to make sense of Pascal's Triangle, he was already speaking in general terms about the shape of the triangle and the meaning of the numbers, Mike anticipated that the next request would be for a general equation; he selected the letter $r$ to stand for a general row of the triangle and $n$ to stand for a general entry in that row. (This is not the way the equation is normally written in textbooks, where the $n$ and $r$ are reversed from the positions that Mike assigned to them.) Mike wrote Equation 3 using the letters he assigned.

Equation 3:

$$\binom{r}{n} + \binom{r}{n+1} = \binom{r+1}{n+1}$$

This is equivalent to equations 1 and 2. It is interesting to note that Mike's notation was internally consistent but different from both what the group had done in high school and from what he might find in a textbook. This suggests that he was not just recalling a memorized formula but that he was using notation that made sense to him. It appears that his sense of the meaning of Pascal's Identity enabled him to recreate its equation in available notation.

**EPISODE 7: INTERVIEW WITH ANKUR**

I interviewed Ankur just before his third year of college. He watched a videotape of Episode 5, and he wrote Pascal's Identity as the students wrote it (Equation 2) as he viewed the discussion. When I asked him to explain the rule for Pascal's Identity, he began with a mention of form; he said he noticed that the bottom number in the result always came from the rightmost number on the left side of the equation. We turned to a specific instance of the addition rule, shown in Equation 4. After he explained that case in terms of towers – one builds the six 4-tall towers with two red cubes by adding a red cube to the three 3-tall towers that have one red cube and by adding a blue cube to the other towers – he went on to explain the general case in terms of towers also. He said that in the general
case you add a red cube to the towers with fewer red cubes (the "$N \text{ choose } X$" case) and you add the other color cube to the towers that already have the required number of red cubes (the "$N \text{ choose } X+1$" case).

Equation 4: \[
\binom{3}{1} + \binom{3}{2} = \binom{4}{2}
\]

Although the explanation that he saw involved pizza toppings, Ankur chose to explain Pascal’s Identity in terms of manipulations on towers. This suggests that his explanation of the rule was not just a restatement of what he had seen on the video. Ankur used a specific set of towers to assist in recalling the general case, and then he gave a meaning of the general equation in terms of building towers in general. Note that his tower description was partly general; although he mentioned a specific color for one case (red), his use of “the other color” indicates an acknowledgement that the specific colors are irrelevant.

**EPISODE 8: INTERVIEW WITH ROMINA**

When I interviewed Romina in the summer before her third year in college, I showed her a copy of Figure 2 (Pascal's Triangle in “choose” notation). I asked if she could explain Pascal’s Identity in terms of towers. Her response referenced both towers and pizzas. A partial transcript follows.

U: Michael … would explain things by talking about, you're going from a number of pizza toppings to a different number of pizza toppings and how the addition rule works there. And … you seemed to think of towers as the primary way to do Pascal's Triangle. … Do you remember anything from how you guys worked on this or how the addition rule would apply here?

Rom.: … I think this is how many toppings. Like the top number, like the one choose one or one choose zero would be how many toppings. Or I mean if we were talking about towers … [Romina points to row 2 of Pascal’s Triangle.] This would be with two high with zero reds, one red, two reds. And it just keeps going like three high, zero reds, one red, two, then three reds. So it would be like three high and like out of those, you choose how many blocks of each color.

In order to ask Romina about Pascal’s Identity, I drew a general row of Pascal’s Triangle. Like Mike, I used nonstandard notation ($R \text{ choose } N$ instead of $N \text{ choose } R$). I began by asking Romina to give the equation for a specific case of Pascal’s Identity (from row 3 to row 4). But Romina went right to the general case and produced equation 5. Her explanation involved pizzas, towers, and binary notation. She noted that the first term could be represented by pizzas that would receive a new topping, towers that would have a red cube added, or binary numbers that would be extended by the addition of a 1. The other term, she
noted, could represented by pizzas that would not get a topping, towers that would have a blue cube added, or binary numbers that would be extended by the addition of a 0.

\[ \binom{R}{N} + \binom{R}{N+1} = \binom{R+1}{N+1} \]

Equation 5: 

Romina worked with the available notation to provide a general form of Pascal’s Identity. She explained the notation in terms of the problems I asked about (pizzas and towers) and also in terms of a general notation (binary numbers).

SUMMARY

Davis and Maher (1997) maintained that students can learn new mathematics by building on powerful representations with which they are already familiar. I have given examples showing that the students in this study built ideas that came from personal experience; their experience with personal representations helped them to do real mathematics in the sense discussed by Davis and Maher. Their personal representations helped them recognize the isomorphism between two problems that on the surface did not appear to be the same. They located familiar numbers (numbers they knew as solutions to the towers and pizza problems) in Pascal’s Triangle, and then they used Pascal’s Triangle to explain how those problems were related. They explained Pascal’s Identity in terms of the rules for generating successive answers to the towers and pizza problems.

When a standard notation was introduced, the students used what they already knew about Pascal’s Triangle to make a connection between their personal representations and the standard notation. They then used the standard notation to express the relationships among the numbers in Pascal’s Triangle. This standard notation gave them a tool for making a general, concise, and mathematically understandable statement that they had previously expressed using the personal representations connected to towers and pizzas.

Their understanding appears durable. When three of the students were interviewed three years later, they continued to demonstrate both an ability to work with the standard notation and an ability to explain its meaning.

REFERENCES


Yackel, & K. McClain (Eds.), *Symbolizing and Communicating in Mathematics Classrooms* (pp. 325-360). Mahwah, NJ: Lawrence Erlbaum Associates.


